Coalitional Games

- A coalitional game with transferable payoffs consists of:
  - a finite set of players N
  - a function v that associates a real number $v(S)$ (value, or worth os S) with every nonempty subset S of N
- A coalitional game is **cohesive** if:

  $$v(N) \geq \sum_{k=1}^{K} v(S_k) \text{ for every partition } \{S_1,\ldots,S_K\} \text{ of } N$$

The Core

- The Core of a coalitional game is the set of payoff profiles ***x*** (N payoff vectors) for which $x(S) \geq v(S)$ for every $S \subset N$ where

$$x(S) = \sum_{i \in S} x_i$$

- Example: Three-player majority game
  - The grand coalition can obtain 1; $v(N) = 1$
  - Each 2 players can together obtain $a \in [0,1]$;
    $v(S) = a$ if |S|=2
  - A single player can obtain nothing; $v(i) = 0$
- In order for x to be in the core, it must be that:
  - $x(N) = 1$
  - $x(S) \geq a$ if |S|=2
  - $x(i) \geq 0$
- You can see that the core is nonempty if $a \leq 2/3$

The Core cont.

- Example: Market for an indivisible good
- B – the set of buyers with valuation 1,
- L – the set of sellers with reservation price 0
- $N = B \cup L$, $v(S) = \min\{|S \cap B|, |S \cap L|\}$
- Suppose that $|B| > |L|$. What is the core?
- Let l and b be the indexes of the seller and the buyer with the lowest x_i. For these 2 agents it must hold that $x_l + x_b \geq v(\{b, l\})=1$
- and $|L| = v(N) = x(N) \geq |B|x_b + |L|x_l =$
  $= |L|\ (x_b + x_l) + |B\text{-}L|x_b \geq |L| + |B\text{-}L|x_b$
- which implies $x_i = 0$ for all buyers and $x_i = 1$ for all sellers

More on the Core

- The idea of the core extends to general coalitional games (without transferable payoffs)
- Then the core is the set of outcomes x such that no coalition can achieve something that is preferred to x by all members of the coalition
- Example: Exchange economy - every competitive allocation in an exchange economy belongs to the core.
- There are many refinements of and solutions alternative to the concept of the core
- They restrict the way in which a coalition is formed or the way it acts

Shapley Value

- The Shapley Value is defined by

$$\phi_i(N,v) = \frac{1}{|N|!} \sum_{R \in \Re} \Delta_i(S_i(R))$$

- for each $i \in N$ where is the set of all $|N|!$ orderings of N, $S_i(R)$ is the set of players preceding i in the ordering R, and

$$\Delta_i(S) = v(S \cup \{i\}) - v(S)$$

- The usual interpretation: suppose that the players will be joining the grand coalition in some randomly selected order, and that each ordering is equally likely. Then the value of player i is his *expected contribution* to the set of players who preceded him.

Shapley Value cont.

- Example. The market for an indivisible good with N=3, |B|=2, |L|=1

Ordering	v. added by 1	v. added by 2	v. added by 3
123	0	1	0
132	0	0	1
213	1	0	0
231	1	0	0
312	1	0	0
321	1	0	0
Expected value added (S.V.)	2/3	1/6	1/6

Shapley Value cont.

- The above shows that S.V. may be outside the core. However, most people would say that S.V. is a better prediction of the allocation in this example than the core. Interestingly: if this game is replicated, the S.V. approaches the core allocation.
- Another nice thing about the core is that it is the only solution that satisfies the following axioms:
  - SYMMETRY: if i and j are interchangeable then $\phi_i(v) = \phi_j(v)$
  - DUMMY: if i is a dummy in v then $\phi_i(v) = v(\{i\})$
  - ADDITIVITY: For any two games v and w we have
    $\phi_i(v+w) = \phi_j(v) + \phi_j(w)$ for all i
    where $v + w$ is the game defined by $(v+w)(S) = v(S) + w(S)$

i and j are interchangeable if $\Delta_i(S) = \Delta_j(S)$
for all S that contain neither i nor j

i is a dummy if $\Delta_i(S) = v_i(\{i\})$
for every coalition S that does not include i

Axiomatic Bargaining

- A *bargaining problem* among I agents consists of 2 elements:
  - the utility possibility set (the bargaining set) $U \subset \boldsymbol{R}^{\boldsymbol{I}}$, U is convex and closed
  - the threat point (status quo) $u^* \in U$ (each agent has veto power)
- A *bargaining solution* is a rule (function) that assigns a vector $f(U, u^*) \in U$ to every bargaining problem (U, u^*).
- Most popular solutions:
  - Egalitarian
  - Utilitarian
  - Nash
  - Kalai-Smorodinsky

Axioms

- **Independence of Utility Origins** (IUO): the bargaining solution is IUO if for any $\alpha = (\alpha_1, ..., \alpha_I) \in \boldsymbol{R}^I$ if for every i we have

$$f_i(U', u^* + \alpha) = f_i(U, u^*) + \alpha_i$$

whenever $U' = \{(u_1 + \alpha_1, ..., u_I + \alpha_I) : u \in U\}$

this property allows us to normalize the problem to $u^* = 0$ and $f(U)$ will denote $f(U, 0)$

- **Independence of Utility Units** (IUU): the bargaining solution is IUU if for any $\beta = (\beta_1, ..., \beta_I) \in \boldsymbol{R}^I$ with $\beta_i > 0$ for every i we have

$$f_i(U') = \beta_i f_i(U)$$

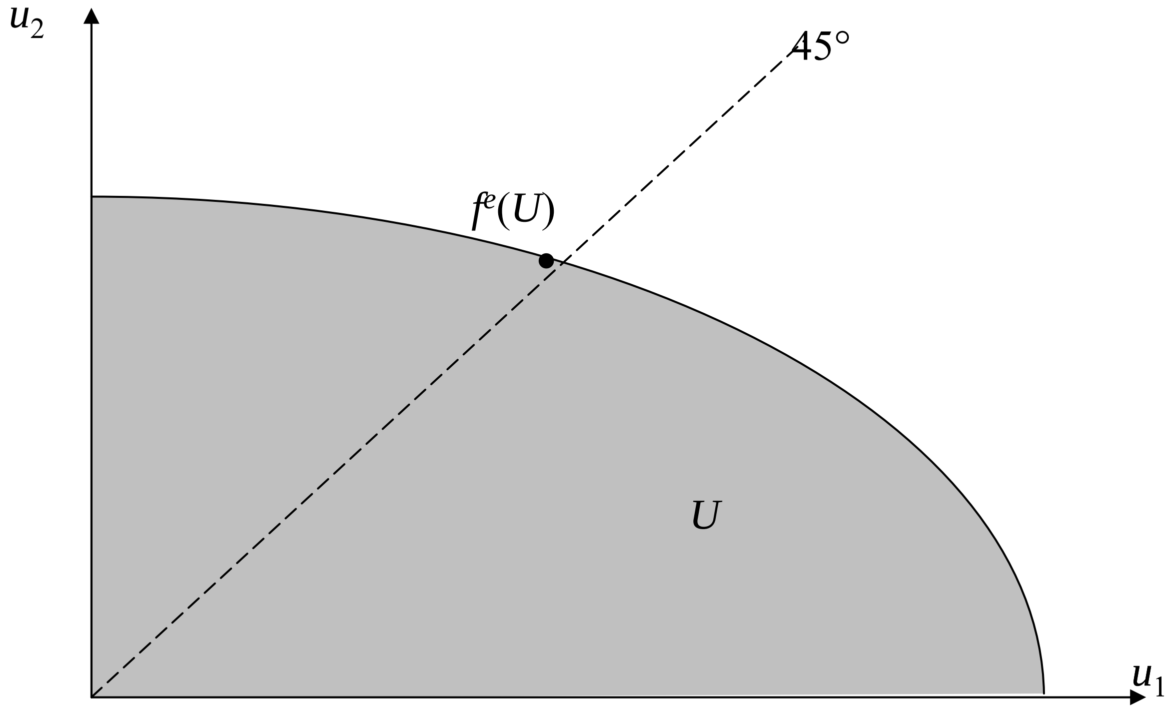
whenever $U' = \{(\beta_1 u_1, ..., \beta_I u_I) : u \in U\}$

Axioms cont.

- **Independence of Irrelevant Alternatives** (IIA): the bargaining solution is IUU if whenever $U' \subset U$ and $f(U) \in U'$, it follows that $f(U') = f(U)$
- **Symmetry** (S): the bargaining solution is symmetric if whenever $U \subset \boldsymbol{R}^{I}$ is a symmetric set, (i.e. U does not change with the permutation of axes, we have that $f_i(U) = f_j(U)$ for any i and j
- **Pareto** (P): the bargaining solution is Pareto if for every U $f(U)$ is a (weak) Pareto optimum, i.e. there is no $u \in U$ such that $u_i > f_j(U)$ for every i
- **Individual Rationality** (IR): the bargaining solution is IR if $f(U) \geq 0$

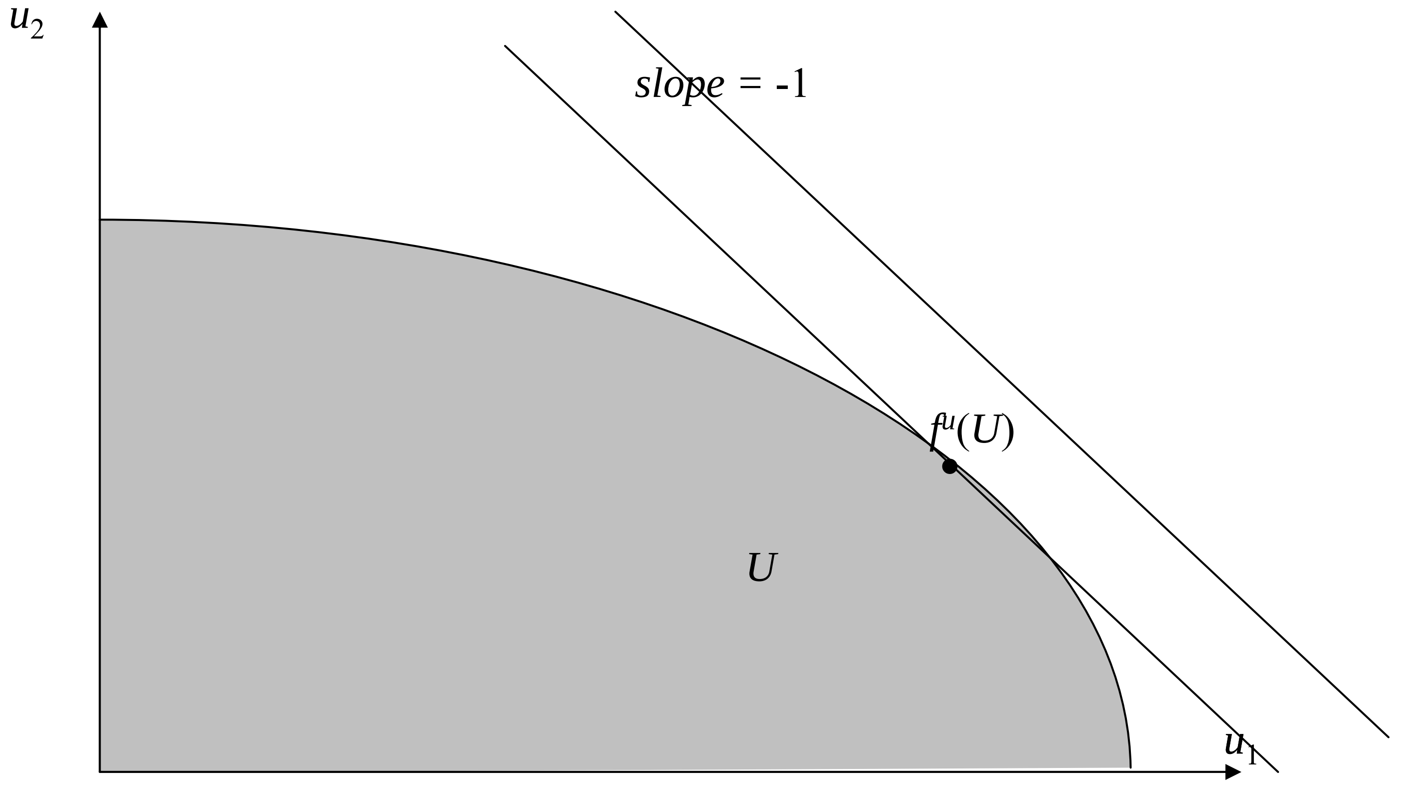
Egalitarian solution

- At the **egalitatian** solution $f^e(.)$, the **gains** from cooperation are split equally among the agents. $f^e(U)$ is a vector in the frontier of U with all entries equal, i.e. $f_i(U) = f_j(U)$ for any i and j. Satisfies IUO, IIA, Symmetry, Pareto, but not IUU.



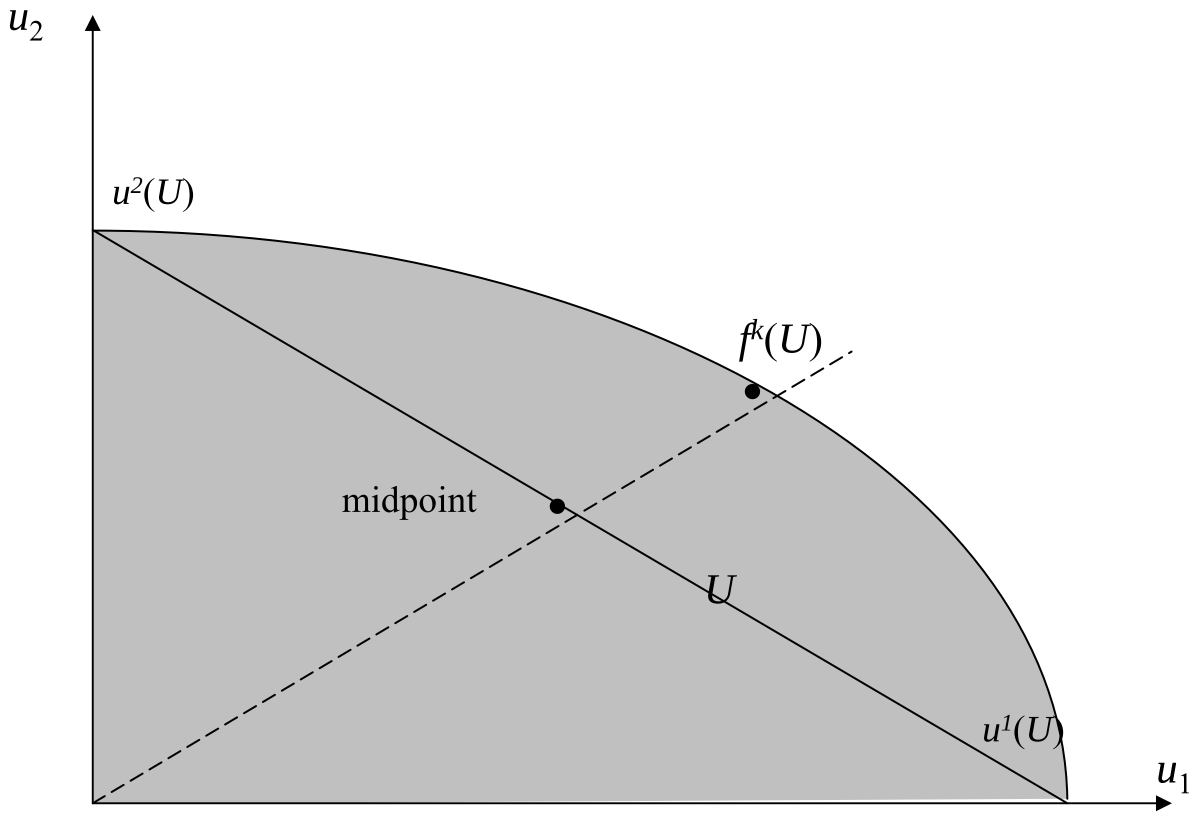
Utilitarian solution

- At the utilitarian solution $f^u(.)$, for every U $f^u(.)$ maximizes $\Sigma_i u_i$ on $U \cap \mathbf{R}^{\mathbf{I}}_+$. Satisfies IUO, IIA, Pareto, Symmetry, IIA (for strictly convex U), but fails IUU



Kalai-Smorodinsky solution

- Let $u^i(U)$ denote the highest utility that agent i could attain in U. The K-S solution $f^k(U)$ is a Pareto optimal allocation with $f^k(U)$ proportional to $(u^1(U), ..., u^I(U))$. K-S satisfies IUO, IUU, Symmetry, Pareto, but fails IIA.



Nash solution

- The Nash solution $f^n(.)$ is a point in U that maximizes the (Nash) product of utilities $u_1 \cdot u_2 \cdot ... \cdot u_I$, or, equivalently, maximizes $\Sigma_i \ln u_i$. Nash solution is **the only** solution that satisfies IUO, IUU, Pareto, Symmetry and IIA.